Characterization of projectively flat Finsler manifolds of constant curvature with finite dimensional holonomy group

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Dedicated to Professor Lajos Tamássy on his 90th birthday

Abstract

In this paper we prove that the holonomy group of a simply connected locally projectively flat Finsler manifold of constant curvature is a finite dimensional Lie group if and only if it is flat or it is Riemannian.

1 Introduction

A Finsler manifold is a pair (M, F), where M is an n-manifold and $F : TM \to \mathbb{R}$ is a non-negative function, smooth and positive away from the zero section of TM, positively homogeneous of degree 1, and strictly convex on each tangent space. A Finsler manifold of dimension 2 is called *Finsler surface*.

The concept of Finsler manifold is a direct generalization of the Riemannian one. The fundamental tensor $g = g_{ij}dx^i \otimes dx^j$ associated to $\mathcal{F}$ is formally analogous to the metric tensor in Riemannian geometry. It is defined by

$$g_{ij} := \frac{1}{2}\frac{\partial^2 \mathcal{F}^2}{\partial y^i \partial y^j}, \tag{1}$$

in an induced standard coordinate system (x, y) on TM. As in Riemannian geometry, a canonical connection Γ can be defined for a Finsler space [4]. However, since the energy function $E = \frac{1}{2}\mathcal{F}^2$ is not necessarily quadratic and only homogeneous, the connection is in general non-linear. In the case, when the connection Γ is linear, the Finsler space is called *Berwald space*. In particular, every Riemannian manifold is a Berwald space.

Due to the existence of the canonical connection, the holonomy group of a Riemannian or Finsler manifold can be defined in a very natural way: it is the group generated by parallel translations along closed curves. In the Riemannian case,

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since the Levi-Civita connection is linear and preserve the Riemannian metric, the holonomy group is a Lie subgroup of the orthogonal group $O(n)$ (see [2]). The Riemannian holonomy theory has been extensively studied, and by now, its complete classification is known.

The holonomy properties of Finsler spaces is essentially different from the Riemannian one. It is proved in [6] that the holonomy group of a Finsler manifold of nonzero constant curvature with dimension greater than 2 is not a compact Lie group. In [8] large families of projectively flat Finsler manifolds of constant curvature are constructed such that their holonomy groups are not finite dimensional Lie groups. There are explicitly given examples of Finsler 2-manifolds having maximal holonomy group. In these examples the closure of the holonomy group is isomorphic to the orientation preserving diffeomorphism group of the 1-dimensional sphere [9].

In this paper we are investigating the holonomy group of locally projectively flat Finsler manifolds of constant curvature. A Finsler function $\mathcal{F}$ on an open subset $D \subset \mathbb{R}^n$ is called *projectively flat*, if all geodesic curves are straight lines in D. The Finsler manifold $(M, \mathcal{F})$ is said to be locally projectively flat, if for any point there exists a local coordinate system in which $\mathcal{F}$ is projectively flat. Our aim is to characterize all locally projectively flat Finsler manifolds with finite dimensional holonomy group. To obtain such a characterization, we will investigate the dimension of the infinitesimal holonomy algebra which was introduced in the Finsler case in [7]. In Proposition 3.2 we prove that if $(M, \mathcal{F})$ is a non-Riemannian locally projectively flat Finsler manifolds of nonzero constant curvature, then its infinitesimal holonomy algebra is infinite dimensional. Using this result and the tangent property of the infinitesimal holonomy algebra proved in [7] we obtain the characterization given by Theorem 3.6: *The holonomy group of a locally projectively flat Finsler manifold of constant curvature is finite dimensional if and only if it is a Riemannian manifold or a flat Finsler manifold.*

2 Preliminaries

Throughout this article, M is a C^∞ smooth simply connected manifold, $\mathfrak{X}^\infty(M)$ is the vector space of smooth vector fields on M and $\mathsf{Diff}^\infty(M)$ is the group of all C^∞-diffeomorphism of M. The first and the second tangent bundles of M are denoted by (TM, π, M) and (TTM, τ, TM), respectively.

2.1 Finsler manifolds

A *Finsler manifold* is a pair $(M, \mathcal{F})$, where the Finsler function $\mathcal{F}\colon TM \to \mathbb{R}$ is a continuous function, smooth on $\hat{T}M := TM \setminus \{0\}$, its restriction $\mathcal{F}_x = \mathcal{F}|_{T_xM}$ is a positively homogeneous function of degree one and the symmetric bilinear form

$$g_{x,y}\colon (u,v) \ \mapsto \ g_{ij}(x,y)u^iv^j = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2_x(y+su+tv)}{\partial s\,\partial t}\Big|_{t=s=0} \tag{2}$$

is positive definit. The Finsler manifold $(M, \mathcal{F})$ is Riemannian, if $\mathcal{F}^2$ induces a quadratic form on any tangent space T_xM. Hence we say that $(M, \mathcal{F})$ is *non-Riemannian Finsler manifold* if there exists a point $x \in M$ such that $\mathcal{F}^2_x$ is not

quadratic.

A vector field $X(t)=X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c(t)$ is said to be parallel with respect to the associated *homogeneous (nonlinear) connection* if it satisfies

$$D_{\dot c}X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t),X(t))\dot c^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \tag{3}$$

where the *geodesic coefficients* $G^i(x,y)$ are given by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^jy^k. \tag{4}$$

and $G^i_j = \frac{\partial G^i}{\partial y^j}$. The *horizontal Berwald covariant derivative* $\nabla_X\xi$ of $\xi(x,y) = \xi^i(x,y)\frac{\partial}{\partial y^i}$ by the vector field $X(x)=X^i(x)\frac{\partial}{\partial x^i}$ is expressed locally by

$$\nabla_X\xi = \left(\frac{\partial \xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial \xi^i(x,y)}{\partial y^k} + G^i_{jk}(x,y)\xi^k(x,y)\right)X^j\frac{\partial}{\partial y^i}, \tag{5}$$

where $G^i_{jk}(x,y) := \frac{\partial G^i_j(x,y)}{\partial y^k}$.

A Finsler manifold $(M,\mathcal{F})$ is said to be *projectively flat*, if there exists a diffeomorphism of M to an open subset $D\subset\mathbb{R}^n$ such that the images of geodesic curves are straight lines in D. A Finsler manifold $(M,\mathcal{F})$ is said to be *locally projectively flat*, if for every $x\in M$ there exists a local coordinate system (U,x) such that $x=(x^1,\dots,x^n)$ is mapping the neighbourhood U into the Euclidean space $\mathbb{R}^n$ such that the straight lines of $\mathbb{R}^n$ correspond to the geodesics of $(M,\mathcal{F})$ on U. Then there exists a function $\mathcal{P}(x,y)$, such that the geodesic coefficients are given by

$$G^i(x,y) = \mathcal{P}(x,y)y^i, \qquad i=1,...,n \tag{6}$$

The function $\mathcal{P}=\mathcal{P}(x,y)$ is called the *projective factor* of $(M,\mathcal{F})$ on U. Since it is 1-homogeneous in the y-variable, we have also the following relations:

$$G^i_k = \frac{\partial\mathcal{P}}{\partial y^k}y^i + \mathcal{P}\delta^i_k, \quad G^i_{kl} = \frac{\partial^2\mathcal{P}}{\partial y^k\partial y^l}y^i + \frac{\partial\mathcal{P}}{\partial y^k}\delta^i_l + \frac{\partial\mathcal{P}}{\partial y^l}\delta^i_k. \tag{7}$$

It follows from equation (6) that the associated homogeneous connection (3) is linear if and only if the projective factor $\mathcal{P}(x,y)$ is linear in y. According to Lemma 8.2.1 in [3] p.155, if $(M\subset\mathbb{R}^n,\mathcal{F})$ is a projectively flat Finsler manifold with constant flag curvature λ, then we have

$$\mathcal{P} = \frac{1}{2\mathcal{F}}\frac{\partial\mathcal{F}}{\partial x^i}y^i, \qquad \mathcal{P}^2 - \frac{\partial\mathcal{P}}{\partial x^i}y^i = \lambda\mathcal{F}^2. \tag{8}$$

Hence if $\lambda\neq 0$ and $\mathcal{P}(x,y)$ is linear in y at $x\in M$ then $\mathcal{F}^2(x,y)$ is a quadratic form in y at x.

2.2 Holonomy group, curvature, infinitesimal holonomy algebra

For a Finsler manifold $(M, \mathcal{F})$ of dimension n the *indicatrix* at $x \in M$ is

$$\mathcal{I}_x M := \{y \in T_x M \mid \ \mathcal{F}(y) = 1\}$$

in $T_x M$ which is an $(n-1)$-dimensional submanifold of $T_x M$. We denote by $(\mathcal{I}M, \pi, M)$ the *indicatrix bundle* of $(M, \mathcal{F})$. We remark that, although the homogeneous (nonlinear) parallel translation is in general not metrical, that is it does not preserve the Finsler metric tensor (2), but it preserves the value of the Finsler function. That means that for any curves $c : [0,1] \to M$, the induced parallel translation $\tau_c : T_{c(0)}M \to T_{c(1)}M$ induces a map $\tau_c \colon \mathcal{I}_{c(0)}M \to \mathcal{I}_{c(1)}M$ between the indicatrices.

The *holonomy group* $\mathsf{Hol}_x(M)$ of $(M, \mathcal{F})$ at a point $x \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_x M)$ generated by (nonlinear) parallel translations of $\mathcal{I}_x M$ along piece-wise differentiable closed curves initiated at the point $x \in M$.

The *Riemannian curvature tensor* $R = R^i_{jk}(x,y) dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$ has the expression

$$R^i_{jk}(x,y) = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y) G^i_{km}(x,y) - G^m_k(x,y) G^i_{jm}(x,y).$$

The manifold has *constant flag curvature* $\lambda \in \mathbb{R}$, if for any $x \in M$ the local expression of the Riemannian curvature is

$$R^i_{jk}(x,y) = \lambda\big(\delta^i_k g_{jm}(x,y) y^m - \delta^i_j g_{km}(x,y) y^m\big).$$

For any vector fields $X, Y \in \mathfrak{X}^\infty(M)$ on M the vector field $\xi = R(X,Y) \in \mathfrak{X}^\infty(\mathcal{I}M)$ is called a *curvature vector field* of $(M, \mathcal{F})$ (see [6]). The Lie algebra $\mathfrak{R}(M)$ of vector fields generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of $(M, \mathcal{F})$. The restriction $\mathfrak{R}_x(M) := \big\{ \xi\big|_{\mathcal{I}_x M} \ ; \ \xi \in \mathfrak{R}(M) \big\} \subset \mathfrak{X}^\infty(\mathcal{I}_x M)$ of the curvature algebra to an indicatrix $\mathcal{I}_x M$ is called the *curvature algebra at the point* $x \in M$.
We remark that the indicatrix of a Finsler surface is 1-dimensional at any point $x \in M$, hence the curvature vector fields at $x \in M$ are proportional to any given non-vanishing curvature vector field. Therefore the curvature algebra $\mathfrak{R}_x(M)$ is at most a 1-dimensional commutative Lie algebra.

The *infinitesimal holonomy algebra* of $(M, \mathcal{F})$ is the smallest Lie algebra $\mathfrak{hol}^*(M)$ of vector fields on the indicatrix bundle $\mathcal{I}M$ containing the curvature algebra and invariant with respedct to the horizontal Berwald covariant differentiation. The $\mathfrak{hol}^*(M)$ is characterized by the following properties:

(i) any curvature vector field ξ belongs to $\mathfrak{hol}^*(M)$,

(ii) if $\xi, \eta \in \mathfrak{hol}^*(M)$ then $[\xi, \eta] \in \mathfrak{hol}^*(M)$,

(iii) if $\xi \in \mathfrak{hol}^*(M)$ and $X \in \mathfrak{X}^\infty(M)$ then $\nabla_X \xi \in \mathfrak{hol}^*(M)$.

The restriction $\mathfrak{hol}^*_x(M) := \big\{ \xi\big|_{\mathcal{I}_x M} \ ; \ \xi \in \mathfrak{hol}^*(M) \big\} \subset \mathfrak{X}^\infty(\mathcal{I}_x M)$ of the infinitesimal holonomy algebra to an indicatrix $\mathcal{I}_x M$ is called the *infinitesimal holonomy algebra at the point* $x \in M$.
Clearly, we have $\mathfrak{R}(M) \subset \mathfrak{hol}^*(M)$ and $\mathfrak{R}_x(M) \subset \mathfrak{hol}^*_x(M)$ for any $x \in M$ (see [7]).

3 Dimension of the holonomy group

Let (M, F) be a Finsler manifold and $x \in M$ an arbitrary point in M. According to Proposition 3 of [7], the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$. Therefore the group generated by the exponential image of the infinitesimal holonomy algebra at $x \in M$ with respect to the exponential map $\exp_x : \mathfrak{X}^\infty(\mathcal{I}_x M) \to \mathsf{Diff}^\infty(\mathcal{I}_x M)$ is a subgroup of the closed holonomy group $\overline{\mathsf{Hol}_x(M)}$ (see Theorem 3.1 of [9]). Consequently, we have the following estimation on the dimensions:

$$\dim \mathfrak{hol}^*_x(M) \leq \dim \mathsf{Hol}_x(M). \tag{9}$$

Using the result of S. Lie claiming that the dimension of a finite-dimensional Lie algebra of vector fields on a connected 1-dimensional manifold is less than 4 (cf. [1], Theorem 4.3.4) we can obtain the following

Lemma 3.1. If the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of a Finsler surface $(M, \mathcal{F})$ contains 4 simultaneously non-vanishing $\mathbb{R}$-linearly independent vector fields, then $\mathfrak{hol}^*_x(M)$ is infinite dimensional.

Proof. If the infinitesimal holonomy algebra is finite-dimensional, then the dimension of the corresponding Lie group acting locally effectively on the 1-dimensional indicatrix would be at least 4, which is a contradiction. □

Using Lemma 3.1 we can prove the following

Proposition 3.2. *The infinitesimal holonomy algebra of any locally projectively flat non-Riemannian Finsler surface* $(M, \mathcal{F})$ *of constant curvature* $\lambda \neq 0$ *is infinite dimensional.*

Proof. Assume that the locally projectively flat Finsler surface $(M, \mathcal{F})$ of non-zero constant curvature λ is non-Riemannian at a fixed point $x \in M$. Let (x^1, x^2) be a local coordinate system centered at x, corresponding to the canonical coordinates of the Euclidean plane which is projectively related to $(M, \mathcal{F})$, and let (y^1, y^2) be the induced coordinate system in the tangent planes $T_x M$.

Consider the curvature vector field

$$\xi(x,y) = R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)(x,y) = \lambda\big(\delta^i_2 g_{1m}(x,y)y^m - \delta^i_1 g_{2m}(x,y)y^m\big)\frac{\partial}{\partial x^i}$$

at the point $x \in M$. Since $(M, \mathcal{F})$ is of constant flag curvature, the horizontal Berwald covariant derivative $\nabla_W R$ of the tensor field R vanishes and one has

$$\nabla_W \xi = R\left(\nabla_k\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right)\right) W^k.$$

Since

$$\nabla_k\left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}\right) = \left(G^1_{k1} + G^2_{k2}\right)\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$$

we obtain $\nabla_W \xi = \left(G^1_{k1} + G^2_{k2}\right) W^k \xi$. According to (7) we have $G^m_{km} = 3\frac{\partial P}{\partial y^k}$ and hence $\nabla_k \xi = 3\frac{\partial P}{\partial y^k}\xi$, where $\nabla_k = \nabla_{\frac{\partial}{\partial x^k}}$. Moreover we have

$$\nabla_j\left(\frac{\partial \mathcal{P}}{\partial y^k}\right) = \frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - G^m_j \frac{\partial^2 \mathcal{P}}{\partial y^m \partial y^k} = \frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - \mathcal{P}\frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^j},$$

and hence

$$\nabla_j(\nabla_k \xi) = 3\left(\frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} - \mathcal{P}\frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^j} + 3\frac{\partial \mathcal{P}}{\partial y^k}\frac{\partial \mathcal{P}}{\partial y^j}\right)\xi.$$

According to Lemma 8.2.1, equation (8.25) in [3], p. 155, we have

$$\frac{\partial^2 \mathcal{P}}{\partial x^j \partial y^k} = \frac{\partial \mathcal{P}}{\partial y^j}\frac{\partial \mathcal{P}}{\partial y^k} + \frac{\partial^2 \mathcal{P}}{\partial y^j \partial y^k} - \lambda\, g_{jk}, \tag{10}$$

hence $\nabla_j(\nabla_k \xi) = 3\left(4\frac{\partial \mathcal{P}}{\partial y^j}\frac{\partial \mathcal{P}}{\partial y^k} - \lambda\, g_{jk}\right)\xi$. It follows the

Lemma 3.3. For any fixed $1 \le j, k \le 2$

$$y \to \xi(x,y), \quad y \to \nabla_1\xi(x,y), \quad y \to \nabla_2\xi(x,y), \quad y \to \nabla_j(\nabla_k\xi)(x,y), \tag{11}$$

considered as vector fields on $\mathcal{I}_x M$, are $\mathbb{R}$-linearly independent if and only if the

$$1, \qquad \frac{\partial \mathcal{P}}{\partial y^1}, \qquad \frac{\partial \mathcal{P}}{\partial y^2}, \qquad \frac{\partial^2 \mathcal{P}}{\partial y^j \partial y^k} - \frac{\lambda}{4}\, g_{jk} \tag{12}$$

are linearly independent functions on $T_x M$.

Since we assumed that the Finsler function $\mathcal{F}$ is non-Riemannian at the point x, then $\mathcal{F}^2(x,y)$ is non-quadratic in y and hence the function $\mathcal{P}(x,y)$ is non-linear in y on $T_x M$ (cf. eq. (8)). Let us choose a direction $y_0 = (y_0^1, y_0^2) \in T_x M$ with $y_0^1 \neq 0$, $y_0^2 \neq 0$ and having property that $\mathcal{P}$ is non-linear 1-homogeneous function in a conic neighbourhood U of y_0 in $T_x M$. By restricting U if it is necessary we can suppose that for any $y \in U$ we have $y^1 \neq 0$, $y^2 \neq 0$.
To avoid confusion between coordinate indexes and exponents, we rename the fiber coordinates of vectors belonging to U by $(u,v) = (y^1, y^2)$. Using the values of $\mathcal{P}$ on U we can define a 1-variable function $f = f(t)$ on an interval $I \subset \mathbb{R}$ by

$$f(t) := \frac{1}{v}\mathcal{P}(x_1, x_2, tv, v). \tag{13}$$

Then we can express $\mathcal{P}$ and its derivatives with f:

$$\begin{aligned} &\mathcal{P} = v\, f(u/v), && \frac{\partial \mathcal{P}}{\partial y^1} = f'(u/v), && \frac{\partial \mathcal{P}}{\partial y^2} = f(u/v) - \frac{u}{v}f'(u/v), \\ &\frac{\partial^2 \mathcal{P}}{\partial y^1 \partial y^1} = \frac{1}{v}f''(u/v), && \frac{\partial^2 \mathcal{P}}{\partial y^1 \partial y^2} = -\frac{u}{v^2}f''(u/v), && \frac{\partial^2 \mathcal{P}}{\partial y^2 \partial y^2} = \frac{u^2}{v^3}f''(u/v). \end{aligned} \tag{14}$$

Lemma 3.4. The functions $1, \frac{\partial \mathcal{P}}{\partial y^1}, \frac{\partial \mathcal{P}}{\partial y^2}$ are linearly independent.

Proof. A nontrivial relation $a+b\frac{\partial\mathcal{P}}{\partial y^1}+c\frac{\partial\mathcal{P}}{\partial y^2}=0$ yields the differential equation $a+bf'+c(f-tf')=0$. It is clear that both b and c cannot be zero. If $c\neq 0$ we get the differential equation

$$\frac{(a+cf)'}{a+cf}=\frac{1}{t-\frac{b}{c}}.$$

The solutions is $f(t)=t-(a+b)/c$ and therefore the corresponding $\mathcal{P}(u,v)=u-v(a+b)/c$ is linear which is a contradiction. If $c=0$, then $b\neq 0$ and $f=-\frac{a}{b}t+K$. The corresponding $\mathcal{P}(u,v)=-\frac{a}{b}u+Kv$ is again linear which is a contradiction. □

Let us assume now, that the infinitesimal holonomy algebra is finite dimensional. We will show that this assumption leads to contradiction which will prove then, that the infinitesimal holonomy algebra is actually infinite dimensional.

Since $\mathcal{I}_xM$ is 1-dimensional, according to the Lemma 3.1, the 4 vector fields in (11) are linearly dependent for any $j,k\in\{1,2\}$. Using Lemma 3.3 we get that the functions

$$1,\quad \mathcal{P}_1,\quad \mathcal{P}_2,\quad \mathcal{P}_j\mathcal{P}_k-\frac{\lambda}{4}g_{jk} \tag{15}$$

($\mathcal{P}_i=\frac{\partial\mathcal{P}}{\partial y^i}$, $\mathcal{P}_{jk}=\frac{\partial^2\mathcal{P}}{\partial y^j\partial y^k}$) are linearly dependent for any $j,k\in\{1,2\}$. From Lemma 3.4 we know, that the first three functions in (15) are linearly independent. Therefore by the assumption, the fourth function must be a linear combination of the first three, that is there exist constants $a_i,b_i,c_i\in\mathbb{R}$, $i=1,2,3$, such that

$$\begin{aligned}
\frac{\lambda}{4}\,g_{11}&=\mathcal{P}_1\mathcal{P}_1+a_1+b_1\mathcal{P}_1+c_1\mathcal{P}_2,\\
\frac{\lambda}{4}\,g_{12}&=\mathcal{P}_1\mathcal{P}_2+a_2+b_2\mathcal{P}_1+c_2\mathcal{P}_2,\\
\frac{\lambda}{4}\,g_{22}&=\mathcal{P}_2\mathcal{P}_2+a_3+b_3\mathcal{P}_1+c_3\mathcal{P}_2.
\end{aligned} \tag{16}$$

Using (1) we get $\partial_1 g_{21}-\partial_2 g_{11}=0$ and $\partial_1 g_{22}-\partial_2 g_{12}=0$ which yield

$$\begin{aligned}
\mathcal{P}_2\mathcal{P}_{11}-\mathcal{P}_1\mathcal{P}_{12}+b_2\mathcal{P}_{11}+(c_2-b_1)\mathcal{P}_{12}-c_1\mathcal{P}_{22}=0,\\
\mathcal{P}_1\mathcal{P}_{22}-\mathcal{P}_2\mathcal{P}_{12}-b_3\mathcal{P}_{11}+(b_2-c_3)\mathcal{P}_{12}+c_2\mathcal{P}_{22}=0.
\end{aligned} \tag{17}$$

Using the expressions (14) we obtain from (17) the equations

$$\begin{aligned}
(f-\frac{u}{v}f')\frac{1}{v}f''+f'\frac{u}{v^2}f''+b_2\frac{1}{v}f''-(c_2-b_1)\frac{u}{v^2}f''-c_1\frac{u^2}{v^3}f''=0,\\
f'\frac{u^2}{v^3}f''+(f-\frac{u}{v}f')\frac{u}{v^2}f''-b_3\frac{1}{v}f''-(b_2-c_3)\frac{u}{v^2}f''+c_2\frac{u^2}{v^3}f''=0.
\end{aligned} \tag{18}$$

Since by the non-linearity of $\mathcal{P}$ on U we have $f''\neq 0$, equations (18) can divide by f''/v and we get

$$\begin{aligned}
f+\ b_2+\ (b_1-c_2)\frac{u}{v}-\ \frac{c_1u^2}{v^2}\ =\ 0\\
\frac{u}{v}f-\ b_3+\ (c_3-b_2)\frac{u}{v}+\ \frac{c_2u^2}{v^2}\ =\ 0.
\end{aligned} \tag{19}$$

for any $t = u/v$ in an interval $I \subset \mathbb{R}$. The solution of this system of quadratic equations for the function f is $f(t) = -c_2 t - b_2$ with $c_1 = b_3 = 0$, $b_1 = 2c_2$, $c_3 = 2b_2$. But this is a contradiction, since we supposed that by the non-linearity of P we have $f'' \neq 0$ on this interval. Hence the functions 1, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_j\mathcal{P}_k - \frac{\lambda}{4}g_{jk}$ can not be linearly dependent for any $j, k \in \{1, 2\}$, from which follows the assertion. $\square$

Remark 3.5. From Proposition 3.2 we get that if $(M, \mathcal{F})$ is non-Riemannian and $\lambda \neq 0$, then the holonomy group has an infinite dimensional tangent algebra.

Indeed, according to Theorem 6.3 in [7] the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$, from which follows the assertion.

Now, we can prove our main result:

Theorem 3.6. *The holonomy group of a locally projectively flat simply connected Finsler manifold $(M, \mathcal{F})$ of constant curvature λ is finite dimensional if and only if $(M, \mathcal{F})$ is Riemannian or $\lambda = 0$.*

Proof. If $(M, \mathcal{F})$ is Riemannian then its holonomy group is a Lie subgroup of the orthogonal group and therefore it is a finite dimensional compact Lie group. If $(M, \mathcal{F})$ has zero curvature, then the horizontal distribution associated to the canonical connection in the tangent bundle is integrable and hence the holonomy group is trivial. If $(M, \mathcal{F})$ is non-Riemannian having non-zero curvature λ, then for each tangent 2-plane $S \subset T_xM$ the manifold M has a totally geodesic submanifold $\widetilde{M} \subset M$ such that $T_x\widetilde{M} = S$. This $\widetilde{M}$ with the induced metric is a locally projectively flat Finsler surface of constant curvature λ. Therefore from Proposition 3.2 we get that $\mathfrak{hol}^*_x(\widetilde{M})$ is infinite dimensional. Moreover, according to Theorem 4.3 in [8], if a Finsler manifold $(M, \mathcal{F})$ has a totally geodesic 2-dimensional submanifold $\widetilde{M}$ such that the infinitesimal holonomy algebra of $\widetilde{M}$ is infinite dimensional, then the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(M)$ of the containing manifold is also infinite dimensional. Using (9) we get that $\mathsf{Hol}_x(M)$ cannot be finite dimensional. Hence the assertion is true. $\square$

We note that there are examples of non-Riemannian type locally projectively flat Finsler manifolds with $\lambda = 0$ curvature, (cf. [5]).

Remark 3.7. In the discussion before the previous theorem, the key condition for the Finsler metric tensor was not the positive definiteness but its non-degenerate property. Therefore Theorem 3.6 can be generalized as follows.

A pair $(M, \mathcal{F})$ is called *semi-Finsler manifold* if in the definition of Finsler manifolds the positive definitness of the Finsler metric tensor is replaced by the nondegenerate property. Then we have

Corollary 3.8. *The holonomy group of a locally projectively flat simply connected semi-Finsler manifold $(M, \mathcal{F})$ of constant curvature λ is finite dimensional if and only if $(M, \mathcal{F})$ is semi-Riemannian or $\lambda = 0$.*

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